

# A generalized Lyapunov's inequality for a fractional boundary value problem

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## Abstract

We prove existence of positive solutions to a nonlinear fractional boundary value problem. Then, under some mild assumptions on the nonlinear term, we obtain a smart generalization of Lyapunov's inequality. The new results are illustrated through examples.

*Key words:* fractional differential equations, Lyapunov's inequality, boundary value problem, positive solutions, Guo–Krasnoselskii fixed point theorem.

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## 1. Introduction

Lyapunov's inequality is an outstanding result in mathematics with many different applications – see [6, 16] and references therein. The result, as proved by Lyapunov in 1907 [11], asserts that if  $q : [a, b] \rightarrow \mathbb{R}$  is a continuous function, then a necessary condition for the boundary value problem

$$\begin{cases} y'' + qy = 0, & a < t < b, \\ y(a) = y(b) = 0 \end{cases} \quad (1)$$

to have a nontrivial solution is given by

$$\int_a^b |q(s)| \, ds > \frac{4}{b-a}. \quad (2)$$

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Lyapunov's inequality (2) has taken many forms, including versions in the context of fractional (noninteger order) calculus, where the second-order derivative in (1) is substituted by a fractional operator of order  $\alpha$ .

**Theorem 1** (See [3]). *Consider the fractional boundary value problem*

$$\begin{cases} {}_aD^\alpha y + qy = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \quad (3)$$

where  ${}_aD^\alpha$  is the (left) Riemann–Liouville derivative of order  $\alpha \in (1, 2]$  and  $q : [a, b] \rightarrow \mathbb{R}$  is a continuous function. If (3) has a nontrivial solution, then

$$\int_a^b |q(s)| ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}. \quad (4)$$

A Lyapunov fractional inequality (4) can also be obtained by considering the fractional derivative in (3) in the sense of Caputo instead of Riemann–Liouville [4]. More recently, Rong and Bai obtained a Lyapunov-type inequality for a fractional differential equation but with fractional boundary conditions [13]. Motivated by [7, 8, 9, 12] and the above results, as well as existence results on positive solutions [1, 2, 10, 17], which are often useful in applications, we focus here on the following boundary value problem:

$$\begin{cases} {}_aD^\alpha y + q(t)f(y) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \quad (5)$$

where  ${}_aD^\alpha$  is the Riemann–Liouville derivative and  $1 < \alpha \leq 2$ . Our first result asserts existence of nontrivial positive solutions to problem (5) (see Theorem 8). Then, under some assumptions on the nonlinear term  $f$ , we get a generalization of inequality (4) (see Theorem 10).

The paper is organized as follows. In Section 2 we recall some notations, definitions and preliminary facts, which are used throughout the work. Our results are given in Section 3: using the Guo–Krasnoselskii fixed point theorem, we establish in Section 3.1 our existence result; then, in Section 3.2, assuming that function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous, concave and nondecreasing, we generalize Lyapunov's inequalities (2) and (4).

## 2. Preliminaries

Let  $C[a, b]$  be the Banach space of all continuous real functions defined on  $[a, b]$  with the norm  $\|u\| = \sup_{t \in [a, b]} |u(t)|$ . By  $L[a, b]$  we denote the space of all real functions, defined on  $[a, b]$ , which are Lebesgue integrable with the norm

$$\|u\|_L = \int_a^b |u(s)| ds.$$

The reader interested in the fractional calculus is referred to [15]. Here we just recall the definition of (left) Riemann–Liouville fractional derivative.

**Definition 2.** *The Riemann–Liouville fractional derivative of order  $\alpha > 0$  of a function  $u : [a, b] \rightarrow \mathbb{R}$  is given by*

$${}_a D^\alpha u(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_a^t \frac{u(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where  $n = [\alpha] + 1$  and  $\Gamma$  denotes the Gamma function.

**Definition 3.** *Let  $X$  be a real Banach space. A nonempty closed convex set  $P \subset X$  is called a cone if it satisfies the following two conditions:*

- (i)  $x \in P, \lambda \geq 0$ , implies  $\lambda x \in P$ ;
- (ii)  $x \in P, -x \in P$ , implies  $x = 0$ .

**Lemma 4** (Jensen’s inequality [14]). *Let  $\mu$  be a positive measure and let  $\Omega$  be a measurable set with  $\mu(\Omega) = 1$ . Let  $I$  be an interval and suppose that  $u$  is a real function in  $L(d\mu)$  with  $u(t) \in I$  for all  $t \in \Omega$ . If  $f$  is convex on  $I$ , then*

$$f\left(\int_{\Omega} u(t) d\mu(t)\right) \leq \int_{\Omega} (f \circ u)(t) d\mu(t). \quad (6)$$

If  $f$  is concave on  $I$ , then the inequality (6) holds with “ $\leq$ ” substituted by “ $\geq$ ”.

**Lemma 5** (Guo–Krasnoselskii fixed point theorem [5]). *Let  $X$  be a Banach space and let  $K \subset X$  be a cone. Assume  $\Omega_1$  and  $\Omega_2$  are bounded open subsets of  $X$  with  $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ , and let  $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$  be a completely continuous operator such that*

- (i)  $\|Tu\| \geq \|u\|$  for any  $u \in K \cap \partial\Omega_1$  and  $\|Tu\| \leq \|u\|$  for any  $u \in K \cap \partial\Omega_2$ ; or
- (ii)  $\|Tu\| \leq \|u\|$  for any  $u \in K \cap \partial\Omega_1$  and  $\|Tu\| \geq \|u\|$  for any  $u \in K \cap \partial\Omega_2$ .

*Then,  $T$  has a fixed point in  $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .*

### 3. Main results

Let us consider the nonlinear fractional boundary value problem (5) and give its integral representation involving a Green function, which was deduced in [3].

**Lemma 6.** *Function  $y$  is a solution to the boundary value problem (5) if, and only if,  $y$  satisfies the integral equation*

$$y(t) = \int_a^b G(t, s) q(s) f(y(s)) ds,$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \leq s \leq t \leq b, \end{cases} \quad (7)$$

is the Green function associated to problem (5).

*Proof.* Similar to the one found in [3].  $\square$

**Lemma 7.** *The Green function  $G$  defined by (7) satisfies the following properties:*

1.  $G(t, s) \geq 0$  for all  $a \leq t, s \leq b$ ;
2.  $\max_{t \in [a, b]} G(t, s) = G(s, s)$ ,  $s \in [a, b]$ ;
3.  $G(s, s)$  has a unique maximum given by

$$\max_{s \in [a, b]} G(s, s) = \frac{(b-a)^{\alpha-1}}{4^{\alpha-1} \Gamma(\alpha)};$$

4. there exists a positive function  $\varphi \in C(a, b)$  such that

$$\min_{t \in [\frac{2a+b}{3}, \frac{2b-a}{3}]} G(t, s) \geq \varphi(s) G(s, s), \quad a < s < b.$$

*Proof.* The first three properties are proved in [3]. Moreover, we know that Green's function  $G(t, s)$  is decreasing with respect to  $t$  for  $s \leq t$  and increasing with respect to  $t$  for  $t \leq s$  [3]. To prove the fourth property, we define the following functions:

$$g_1(t, s) = \frac{1}{\Gamma(\alpha)} \left[ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1} \right]$$

and

$$g_2(t, s) = \frac{1}{\Gamma(\alpha)} \left[ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-1}}{(b-a)^{\alpha-1}} \right].$$

Obviously,  $G(t, s) > 0$  for  $t, s \in (a, b)$  and one can seek the minimum in an interval of the form  $[a + \frac{b-a}{n}, b - \frac{b-a}{n}]$ , where  $n \geq 3$  is a natural number. For  $t \in [\frac{2a+b}{3}, \frac{2b-a}{3}]$ ,

$$\begin{aligned} \min_{t \in [\frac{2a+b}{3}, \frac{2b-a}{3}]} G(t, s) &= \begin{cases} g_1\left(\frac{2b-a}{3}, s\right) & \text{if } s \in (a, \frac{2a+b}{3}], \\ \min \left\{ g_1\left(\frac{2b-a}{3}, s\right), g_2\left(\frac{2a+b}{3}, s\right) \right\} & \text{if } s \in [\frac{2a+b}{3}, \frac{2b-a}{3}], \\ g_2\left(\frac{2a+b}{3}, s\right) & \text{if } s \in [\frac{2b-a}{3}, b), \end{cases} \\ &= \begin{cases} g_1\left(\frac{2b-a}{3}, s\right) & \text{if } s \in (a, \lambda], \\ g_2\left(\frac{2a+b}{3}, s\right) & \text{if } s \in [\lambda, b), \end{cases} \\ &= \frac{1}{\Gamma(\alpha)} \begin{cases} \left( \frac{(2b-4a)(b-s)}{3(b-a)} \right)^{\alpha-1} - \left( \frac{2b-a}{3} - s \right)^{\alpha-1} & \text{if } s \in (a, \lambda], \\ \left( \frac{b-s}{3} \right)^{\alpha-1} & \text{if } s \in [\lambda, b), \end{cases} \end{aligned}$$

where  $\frac{2a+b}{3} < \lambda < \frac{2b-a}{3}$  is the unique solution of equation

$$g_1\left(\frac{2b-a}{3}, s\right) = g_2\left(\frac{2a+b}{3}, s\right).$$

Set

$$\varphi(s) = \begin{cases} \frac{\left(\frac{(2b-4a)(b-s)}{3}\right)^{\alpha-1} - (b-a)^{\alpha-1} \left(\frac{2b-a}{3} - s\right)^{\alpha-1}}{\left((s-a)(b-s)\right)^{\alpha-1}} & \text{if } s \in (a, \lambda], \\ \left(\frac{(b-a)}{3(s-a)}\right)^{\alpha-1} & \text{if } s \in [\lambda, b]. \end{cases}$$

The proof is complete.  $\square$

Let  $X = C[a, b]$  and define the operator  $T : X \rightarrow X$  as follows:

$$Ty(t) = \int_a^b G(t, s)q(s)f(y(s))ds, \quad y \in X. \quad (8)$$

To prove existence of solution to the fractional boundary value problem (5) it suffices to prove that the map  $T$  has a fixed point in  $K$ .

### 3.1. Existence of positive solutions

To prove existence of nontrivial positive solutions to the fractional boundary value problem (5) we consider the following hypotheses:

$$(H_1) \quad f(y) \geq \gamma^* r_1 \text{ for } y \in [0, r_1],$$

$$(H_2) \quad f(y) \leq \gamma r_2 \text{ for } y \in [0, r_2],$$

where  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous. In what follows we take

$$\gamma := \left( \int_a^b G(s, s)q(s)ds \right)^{-1} \quad \text{and} \quad \gamma^* := \left( \int_{\frac{2a+b}{3}}^{\frac{2b-a}{3}} G(s, s)\varphi(s)q(s)ds \right)^{-1}.$$

**Theorem 8.** *Let  $q : [a, b] \rightarrow \mathbb{R}_+$  be a nontrivial Lebesgue integrable function. Assume that there exist two positive constants  $r_2 > r_1 > 0$  such that the assumptions  $(H_1)$  and  $(H_2)$  are satisfied. Then the fractional boundary value problem (5) has at least one nontrivial positive solution  $y$  belonging to  $X$  such that  $r_1 \leq \|y\| \leq r_2$ .*

For the proof of Theorem 8 we use Lemma 5 with the cone  $K$  given by

$$K := \{y \in X : y(t) \geq 0, a \leq t \leq b\}.$$

*Proof of Theorem 8.* Using the Ascoli–Arzela theorem, we prove that  $T : K \rightarrow K$  is a completely continuous operator. Let

$$\Omega_i = \{y \in K : \|y\| \leq r_i\}.$$

From  $(H_1)$  and Lemma 7, we have for  $t \in [\frac{2a+b}{3}, \frac{2b-a}{3}]$  and  $y \in K \cap \partial\Omega_1$  that

$$\begin{aligned} (Ty)(t) &\geq \int_a^b \min_{t \in [\frac{2a+b}{3}, \frac{2b-a}{3}]} G(t, s) q(s) f(y(s)) ds \\ &\geq \gamma^* \left( \int_a^b G(s, s) \varphi(s) q(s) ds \right) r_1 \\ &\geq \gamma^* \left( \int_{\frac{2a+b}{3}}^{\frac{2b-a}{3}} G(s, s) \varphi(s) q(s) ds \right) r_1 = \|y\|. \end{aligned}$$

Thus,  $\|Ty\| \geq \|y\|$  for  $y \in K \cap \partial\Omega_1$ . Let us now prove that  $\|Ty\| \leq \|y\|$  for all  $y \in K \cap \partial\Omega_2$ . From  $(H_2)$ , it follows that

$$\|Ty\| = \max_{t \in [a, b]} \int_a^b G(t, s) q(s) f(y(s)) ds \leq \gamma \left( \int_a^b G(s, s) q(s) ds \right) r_2 = \|y\|$$

for  $y \in K \cap \partial\Omega_2$ . Thus, from Lemma 5, we conclude that the operator  $T$  defined by (8) has a fixed point in  $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ . Therefore, the fractional boundary problem (5) has at least one positive solution  $y$  belonging to  $X$  such that  $r_1 \leq \|y\| \leq r_2$ .  $\square$

*Example 9.* Consider the following fractional boundary value problem:

$$\begin{cases} {}_0D^{3/2}y + te^y = 0 & \text{if } 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \quad (9)$$

Firstly, let us calculate the values of  $\gamma$  and  $\gamma^*$ . Here,

$$\varphi(s) = \begin{cases} \frac{\sqrt{\frac{2(1-s)}{3}} - \sqrt{\frac{2}{3} - s}}{\sqrt{s(1-s)}} & \text{if } s \in (0, \lambda], \\ \frac{1}{\sqrt{3s}} & \text{if } s \in [\lambda, 1), \end{cases}$$

where  $\lambda \simeq 0.64645$ . Hence, by a simple computation, we get

$$\gamma^* \simeq 26.459 \text{ and } \gamma \simeq 4.514. \quad (10)$$

We choose  $r_1 = \frac{1}{27}$  and  $r_2 = 1$ . Then we get

1.  $f(y) = e^y \geq \gamma^* r_1$  for  $y \in [0, \frac{1}{27}]$ ;
2.  $f(y) = e^y \leq \gamma r_2$  for  $y \in [0, 1]$ .

Therefore, from Theorem 8, problem (9) has at least one nontrivial solution  $y$  in  $X$  such that  $\frac{1}{27} \leq \|y\| \leq 1$ .

### 3.2. Generalized Lyapunov's inequality

The next result generalizes Theorem 1: choosing  $f(y) = y$  in Theorem 10, inequality (11) reduces to (4). Note that  $f \in C(\mathbb{R}_+, \mathbb{R}_+)$  is a concave and nondecreasing function.

**Theorem 10.** *Let  $q : [a, b] \rightarrow \mathbb{R}$  be a real nontrivial Lebesgue integrable function. Assume that  $f \in C(\mathbb{R}_+, \mathbb{R}_+)$  is a concave and nondecreasing function. If the fractional boundary value problem (5) has a nontrivial solution  $y$ , then*

$$\int_a^b |q(t)| dt > \frac{4^{\alpha-1} \Gamma(\alpha) \eta}{(b-a)^{\alpha-1} f(\eta)}, \quad (11)$$

where  $\eta = \max_{t \in [a, b]} y(t)$ .

*Proof.* We begin by using Lemma 6. We have

$$\begin{aligned} |y(t)| &\leq \int_a^b G(t, s) |q(s)| f(y(s)) ds, \\ \|y\| &\leq \int_a^b G(s, s) |q(s)| f(y(s)) ds < \frac{(b-a)^{\alpha-1}}{4^{\alpha-1} \Gamma(\alpha)} \int_a^b |q(s)| f(y(s)) ds. \end{aligned}$$

Using Jensen's inequality (6), and taking into account that  $f$  is concave and nondecreasing, we get that

$$\|y\| < \frac{(b-a)^{\alpha-1} \|q\|_L}{4^{\alpha-1} \Gamma(\alpha)} \int_a^b \frac{|q(s)| f(y(s)) ds}{\|q\|_L} < \frac{(b-a)^{\alpha-1} \|q\|_L}{4^{\alpha-1} \Gamma(\alpha)} f(\eta),$$

where  $\eta = \max_{t \in [a, b]} y(t)$ . Thus,

$$\int_a^b |q(s)| ds > \frac{4^{\alpha-1} \Gamma(\alpha) \eta}{(b-a)^{\alpha-1} f(\eta)}.$$

This concludes the proof.  $\square$

**Corollary 11.** *Consider the fractional boundary value problem*

$$\begin{cases} {}_a D^\alpha y + q(t) f(y) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases}$$

where  $f \in C(\mathbb{R}_+, \mathbb{R}_+)$  is concave and nondecreasing and  $q \in L([a, b], \mathbb{R}_+^*)$ . If there exist two positive constants  $r_2 > r_1 > 0$  such that  $f(y) \geq \gamma^* r_1$  for  $y \in [0, r_1]$  and  $f(y) \leq \gamma r_2$  for  $y \in [0, r_2]$ , then

$$\int_a^b q(t) dt > \frac{4^{\alpha-1} \Gamma(\alpha) r_1}{(b-a)^{\alpha-1} f(r_2)}.$$

*Example 12.* Consider the following fractional boundary value problem:

$$\begin{cases} {}_0D^{3/2}y + t \ln(2+y) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases}$$

We have that

- (i)  $f(y) = \ln(2+y) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous, concave and nondecreasing;
- (ii)  $q(t) = t : [0, 1] \rightarrow \mathbb{R}_+$  is a Lebesgue integral function with  $\|q\|_L = 1 > 0$ .

We computed the values of  $\gamma$  and  $\gamma^*$  in (10). Choosing  $r_1 = 1/40$  and  $r_2 = 1$ , we get

1.  $f(y) = \ln(2+y) \geq \gamma^* r_1$  for  $y \in [0, 1/40]$ ;
2.  $f(y) = \ln(2+y) \leq \gamma r_2$  for  $y \in [0, 1]$ .

Therefore, from Corollary 11, we get that

$$\int_0^1 q(t) dt > \frac{4^{\alpha-1} \Gamma(\alpha) r_1}{(b-a)^{\alpha-1} f(r_2)} \simeq 4.0334 \times 10^{-2}.$$

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